STRONG CONCISENESS OF COPRIME COMMUTATORS IN PROFINITE GROUPS

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Abstract. Let G be a profinite group. The coprime commutators γ_j^* and δ_j^* are defined as follows. Every element of G is both a γ_1^*-value and a δ_0^*-value. For $j \geq 2$, let X be the set of all elements of G that are powers of γ_{j-1}^*-values. An element a is a γ_j^*-value if there exist $x \in X$ and $g \in G$ such that $a = [x, g]$ and $(|x|, |g|) = 1$. For $j \geq 1$, let Y be the set of all elements of G that are powers of δ_{j-1}^*-values. The element a is a δ_j^*-value if there exist $x, y \in Y$ such that $a = [x, y]$ and $(|x|, |y|) = 1$.

In this paper we establish the following results.

A profinite group G is finite-by-pronilpotent if and only if there is k such that the set of γ_k^*-values in G has cardinality less than $2^{\aleph_0}$ (Theorem 1.1).

A profinite group G is finite-by-(prosoluble of Fitting height at most k) if and only if there is k such that the set of δ_k^*-values in G has cardinality less than $2^{\aleph_0}$ (Theorem 1.2).

1. Introduction

A group word w is called concise in the class of groups $\mathcal{C}$ if the verbal subgroup $w(G)$ is finite whenever w takes only finitely many values in a group $G \in \mathcal{C}$. In the sixties Hall raised the problem whether every word is concise in the class of all groups but in 1989 S. Ivanov [17] solved the problem in the negative (see also [23, p. 439]). On the other hand, the problem for residually finite groups remains open (cf. Segal [24, p. 15] or Jaikin-Zapirain [18]). In recent years several new positive results with respect to this problem were obtained (see [2, 15, 14, 6, 8, 7, 11]).

A natural variation of the notion of conciseness for profinite groups was introduced in [9]: the word w is strongly concise in a class of profinite groups $\mathcal{C}$ if the verbal subgroup $w(G)$ is finite in any group $G \in \mathcal{C}$ in which w takes less than $2^{\aleph_0}$ values. Here and throughout the paper, whenever G is a profinite group we write $w(G)$ to denote the *closed* subgroup generated by w-values. A number of new results on strong conciseness of group words can be found in [9, 13, 3, 20].

It was noted in [10] that the concept of (strong) conciseness can be applied in a wider context. Suppose $\mathcal{C}$ is a class of profinite groups and $\phi(G)$ is a subset of G for every $G \in \mathcal{C}$. Is the subgroup generated by $\phi(G)$ finite whenever $|\phi(G)| < 2^{\aleph_0}$? This question is interesting whenever $\phi(G)$ is defined in some natural way and/or properties of the subgroup $\langle \phi(G) \rangle$ have strong impact on the structure of G. It

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was shown in [10] that the set of coprime commutators in a profinite group is strongly concise.

Given a profinite group G and an element $x \in G$, we denote by $|x|$ (respectively $|G|$) the order of x (respectively G) as a supernatural number (or Stenitz number), and $\pi(x)$ (respectively $\pi(G)$) will stand for the set of prime numbers dividing $|x|$ (respectively $|G|$). An element $a \in G$ is a coprime commutator if there are $x, y \in G$ such that $a = [x, y]$ and $(|x|, |y|) = 1$. Here, as usual, $[x, y]$ stands for $x^{-1}y^{-1}x, y$. It is well-known that the set of coprime commutators in a profinite group G generates the nilpotent residual $\gamma_\infty(G)$, that is, the smallest normal subgroup N such that G/N is pronilpotent. Of course, $\gamma_\infty(G) = \cap_i \gamma_i(G)$ is the intersection of the lower central series of G.

Coprime commutators of higher order in finite groups were defined in [25]. The definition naturally extends to the profinite case. Let G be a profinite group. Every element of G is both a γ_1^*-value and a δ_0^*-value. For $j \geq 2$, let X be the set of all elements of G that are powers (in the profinite sense) of γ_{j-1}^*-values. An element a is a γ_j^*-value if there exist $x \in X$ and $g \in G$ such that $a = [x, g]$ and $(|x|, |g|) = 1$. For $j \geq 1$, let Y be the set of all elements of G that are powers (in the profinite sense) of δ_{j-1}^*-values. The element a is a δ_j^*-value if there exist $x, y \in Y$ such that $a = [x, y]$ and $(|x|, |y|) = 1$.

Set $D_1(G) = \gamma_\infty(G)$ and $D_i(G) = \gamma_\infty(D_{i-1})$ for $i = 2, 3, \ldots$. It was shown in [25] that for any j we have the equalities $\gamma_j^*(G) = D_1(G)$ and $\delta_j^*(G) = D_j(G)$. Thus, $\gamma_j^*(G) = 1$ if and only if G is pronilpotent while $\delta_j^*(G) = 1$ if and only if G is prosoluble and has Fitting height at most j (we say that a profinite group has Fitting height at most h if it has a normal series of length h all of whose factors are pronilpotent).

The main results in this paper are as follows.

Theorem 1.1. *A profinite group G is finite-by-pronilpotent if and only if there is k such that the set of γ_k^*-values in G has cardinality less than $2^{\aleph_0}$.*

Theorem 1.2. *A profinite group G is finite-by-(prosoluble of Fitting height at most k) if and only if there is k such that the set of δ_k^*-values in G has cardinality less than $2^{\aleph_0}$.*

In the case where the set of γ_k^*-values (respectively, δ_j^*-values) in a profinite group is finite of size m the subgroup generated by the set is finite of order bounded in terms of m only. This was established for finite groups in [1] and can be extended to profinite groups in a straightforward manner.

2. Preliminaries

For a subgroup H of a profinite group G, we denote by H^G the normal closure of H in G, that is, the minimal normal subgroup of G containing H. For an element $g \in G$ we denote by g^G the conjugacy class of g in G. We use the left-normed notation for commutators; for example, $[x, y, z] = [[x, y], z]$. If X and Y are subsets of G, then $[X, Y]$ stands for the subgroup generated by commutators $[x, y]$, where $x \in X$ and $y \in Y$. If $X = \{x\}$, then we just write $[x, Y]$. If U is an open normal subgroup of G, we write $U \trianglelefteq_o G$. All subgroups of profinite groups are assumed to be closed.

One can easily see that if N is a normal subgroup of G and x an element whose image in G/N is a γ_j^*-value (respectively, a δ_j^*-value), then there exists a γ_j^*-value

y in G (respectively, a δ_j^*-value) such that $x \in yN$. This observation will be used throughout the paper without explicit references.

We begin by quoting a theorem that was already mentioned in the introduction. It was established in [25] for finite groups. The variant for profinite groups is an obvious modification of the finite case.

Theorem 2.1 ([25] Theorems 2.1, 2.7)**.** *Let G be a profinite group. The subgroup $\gamma_k^*(G)$ is trivial if and only if G is pronilpotent. The subgroup $\delta_k^*(G)$ is trivial if and only if G is prosoluble of Fitting height at most k.*

The following results will be helpful.

Proposition 2.2 ([9] Lemma 2.1)**.** *Let $\varphi : X \to Y$ be a continuous map between two non-empty profinite spaces that is nowhere locally constant (i.e. there is no non-empty open subset $U \subseteq_o X$ where $\varphi|_U$ is constant). Then $|\varphi(X)| \geq 2^{\aleph_0}$.*

Lemma 2.3 ([9] Lemma 2.2)**.** *Let G be a profinite group and $g \in G$ be an element whose conjugacy class g^G contains less than $2^{\aleph_0}$ elements. Then g^G is finite.*

It is well-known that if A is a group of automorphisms of a finite group G with $(|A|, |G|) = 1$, then $[G, A] = [G, A, A]$. The following lemma is a stronger version of this result for the case where G is a pronilpotent group.

Lemma 2.4 ([19] Lemma 4.6)**.** *Let φ be a coprime automorphism of a pronilpotent group G. Then the restriction of the mapping*

$$\theta : x \to [x, \varphi]$$

to the set $S = \{[g, \varphi] | g \in G\}$ is bijective.

The original version of Lemma 2.4 states that the continuous map θ is injective when restricted to the set S. However, since S is closed and $\theta(S) \subseteq S$, the surjectivity of θ follows immediately.

The following is a profinite version of Lemma 2.4 in [25]. It will be used together with Lemma 2.4 in order to show that certain elements of a group G are δ_k^*-values.

Lemma 2.5. *Let G be a profinite group and $g_1, \ldots, g_k$ be δ_{k-1}^*-values in G. Suppose that $g_1, \ldots, g_k \in N_G(H)$ for a subgroup $H \leq G$ with $(|H|, |g_i|) = 1$ for every $i \in \{1, \ldots, k\}$. Then, for every $h \in H$, the element $[h, g_1, \ldots, g_k]$ is a δ_k^*-value.*

The next result is a profinite version of Lemma 2.4 in [1]. By a meta-pronilpotent group we mean a profinite group G having a normal pronilpotent subgroup N such that G/N is pronilpotent.

Lemma 2.6. *Let G be a meta-pronilpotent group. Then $\gamma_\infty(G) = \prod_p [K_p, H_{p'}]$, where K_p is a Sylow p-subgroup of $\gamma_\infty(G)$ and $H_{p'}$ is a Hall p'-subgroup of G.*

For a general group word w, the set G_w of w-values of a profinite group G is always closed in G. We will show that the same is true for the sets of γ_k^* and δ_k^*-values.

Proposition 2.7. *Let $S_1, \ldots, S_k$ be closed subsets of a profinite group G. Then the set*

$$C = \{(g_1, \ldots, g_k) \in S_1 \times \cdots \times S_k \mid (|g_i|, |g_{i+1}|) = 1 \text{ for all } i = 1, \ldots, k-1\}$$

is closed in $S_1 \times \cdots \times S_k$. Furthermore, the sets $G_{\gamma_k^*}$ and $G_{\delta_k^*}$ are closed in G.

Proof. Let $\mathcal{P}$ be the set of all primes and $p \in \mathcal{P}$. First notice that for every closed subset S of G the set

$$S_{p'} = \{g \in S \mid p \notin \pi(g)\}$$

is closed since $S_{p'} = \cap_{N \trianglelefteq_o G} S_{p'}N$. Also, the set $S^{\widehat{\mathbb{Z}}} = \{g^\lambda \mid g \in S, \lambda \in \widehat{\mathbb{Z}}\}$ is the image under the continuous map $f(g, \lambda) = g^\lambda$ of the compact set $S \times \widehat{\mathbb{Z}}$, so it is closed too.

Let now A, B be subsets of G. We claim that the set

$$R_{A,B} = \bigcap_{p \in \mathcal{P}} \big((A \times B_{p'}) \cup (A_{p'} \times B)\big) \tag{2.1}$$

is exactly the set of elements $(a, b) \in A \times B$ with $|a|$ and $|b|$ coprime. On the one hand, if $|a|$ and $|b|$ are coprime then $(a, b) \in (A \times B_{p'}) \cup (A_{p'} \times B)$ for every $p \in \mathcal{P}$, because, if $b \in B \setminus B_{p'}$, then $a \in A_{p'}$ necessarily. On the other hand, if $(a, b) \in R_{A,B}$ and a prime p divides $|a|$, then $(a, b) \in (A, B_{p'})$ so p does not divide $|b|$, and the claim follows. Notice now that if A and B are closed, the set $R_{A,B}$ is an intersection of closed subsets of $G \times G$ so it is closed too.

It is now easy to prove by induction on k that the sets $G_{\gamma_k^*}$, $G_{\delta_k^*}$ are closed: just note that $G_{\gamma_k^*}$ is exactly the set $R_{A,B}$ in (2.1) with $A = G_{\gamma_{k-1}^*}^{\widehat{\mathbb{Z}}}$, $B = G$, whereas $G_{\delta_k^*}$ is the set $R_{A,B}$ in (2.1) with $A = B = G_{\delta_{k-1}^*}^{\widehat{\mathbb{Z}}}$.

To prove that the set C is closed in $S_1 \times \cdots \times S_k$, it suffices to notice that by the above arguments the set

$$C_i = S_1 \times \cdots \times S_{i-1} \times R_{S_i, S_{i+1}} \times S_{i+2} \times \cdots \times S_k$$

is closed for every $i \in \{1, \ldots, k-1\}$ and $C = \bigcap_{i=1}^{k-1} C_i$. □

3. Coprime commutators in prosoluble groups

In this section we will some particular cases of Theorems 1.1 and 1.2.

First we will prove Theorem 1.1 in the case when the profinite group G is meta-pronilpotent. The second part of the section will be devoted to the proof of Theorem 1.2 in the case where G is poly-pronilpotent.

We first start with a general reduction.

Lemma 3.1. *Let ϕ be a map that associates to every group G a normal subset $\phi(G) \subseteq G$. Let G be a profinite group with $|\phi(G)| < 2^{\aleph_0}$ and let K be a pronilpotent subgroup of $\langle \phi(G) \rangle$ generated by a subset of $\phi(G)$. If K/K' is finite, then K is finite.*

Proof. Since K is pronilpotent, $\Phi(K) \leq K'$, where $\Phi(K)$ stands for the Frattini subgroup of K. Thus $K/\Phi(K)$ is finite, and hence we can find a finite subset S of $\phi(G)$ generating K. Since $\phi(G)$ is a normal subset of G, by Lemma 2.3 each of these generators has finitely many conjugates in G, so in particular $|G : C_G(s)| \leq \infty$ for every $s \in S$. Since $C_G(K) = \cap_{s \in S} C_G(s)$, this implies that $Z(K) = K \cap C_G(K)$ has finite index in K, and by Schur's theorem K' is finite. □

We will use Lemma 3.1 for $\phi = \gamma_k^*$ or $\phi = \delta_k^*$, but it could be applied to other cases, such as group word maps or uniform (anti-coprime) commutators (see [10] or [12]).

3.1. **The meta-pronilpotent case for γ_k^*.** The next lemma is given without a proof since this is an obvious modification of Lemma 3.1 in [10].

Lemma 3.2. *Let G be a profinite group that is a product of a subgroup H and a normal pronilpotent subgroup Q with $(|H|,|Q|)=1$. Suppose that*

$$|\{[h,q] \mid h\in H, q\in Q\}| < 2^{\aleph_0}.$$

Then $[H,Q]$ is finite.

Similarly, by using Lemma 3.2 in place of Lemma 3.1 of [10], we can refine the proof of Lemma 3.4 of [10] and obtain the following version.

Lemma 3.3. *Let G be a meta-pronilpotent group with*

$$|\{[g,h] \mid g\in G, h\in \gamma_\infty(G), (|g|,|h|)=1\}| < 2^{\aleph_0}.$$

Then $\gamma_\infty(G)$ is finite.

We are now ready to prove the strong conciseness of γ_k^* in meta-pronilpotent groups.

Proposition 3.4. *Let G be a meta-pronilpotent group with $|G_{\gamma_k^*}| < 2^{\aleph_0}$. Then $\gamma_\infty(G)$ is finite.*

Proof. Let $g\in G$ and $h\in\gamma_\infty(G)$ such that $(|g|,|h|)=1$, and let H be the minimal Hall subgroup of $\gamma_\infty(G)$ containing h. Since H is pronilpotent, Lemma 2.4 shows that there exists $h'\in H$ such that $[h,g]=[h',g,\overset{k-1}{\ldots},g]$, and therefore $[h,g]\in G_{\gamma_k^*(G)}$. Hence, we have

$$|\{[g,h] \mid g\in G, h\in \gamma_\infty(G), (|g|,|h|)=1\}| < 2^{\aleph_0},$$

and we conclude by Lemma 3.3. □

3.2. **The poly-pronilpotent case for δ_k^*.** Recall that a Sylow basis of a profinite group G is a family $\{P_i\}$ of Sylow subgroups of G, one for each prime in $\pi(G)$, such that $P_iP_j = P_jP_i$ for every i,j. The normalizer of a Sylow basis, namely $T=\bigcap_i N_G(P_i)$, is pronilpotent and satisfies $G=T\gamma_\infty(G)$. Basic results on Sylow bases can be found in Section 9.2 of [21]. It is well-known that any prosoluble group admits a Sylow basis (see Proposition 2.3.9 of [22]). Moreover, if G is meta-pronilpotent, it is easy to see that $\gamma_\infty(G)=[T,\gamma_\infty(G)]$.

For every profinite group G, the subgroup $\delta_k^*(G)$ coincides with the nilpotent residual $\gamma_\infty(\delta_{k-1}^*(G))$ of $\delta_{k-1}^*(G)$ (see [25]). Let $\{P_i\}$ be a Sylow basis of G and observe that $\{P_i\cap\delta_j^*(G)\}$ is a Sylow basis of $\delta_j^*(G)$ for every $j\geq 1$. Let T_j be the normalizer in $\delta_j^*(G)$ of the Sylow basis $\{P_i\cap\delta_j^*(G)\}$, so that $G=T_1\cdots T_k\delta_{k+1}(G)$ for every $k\geq 1$ and $T_j\leq N_G(T_i)$ for every $j\leq i$. In particular, if G is a prosoluble group of Fitting height k, then $\delta_{k+1}^*(G)=1$, and therefore $G=T_1\cdots T_k$. In this case, we have $T_k=\delta_k^*(G)$ and $T_i=[T_1,\ldots,T_i]$ for all $i\in\{1,\ldots,k\}$.

For this reason, we will often work with subgroups $G_1,\ldots,G_k$ of a profinite group G such that $G_j\leq N_G(G_i)$ for every $j\leq i$; in this setting, we will write

$$\begin{array}{rccc}\varphi: & G_1\times\cdots\times G_k & \longrightarrow & G_k\\ & (g_1,\ldots,g_k) & \longmapsto & [g_1,\ldots,g_k].\end{array}$$

Let

$$T=\{(g_1,...,g_k)\in G_1\times\cdots\times G_k \mid (|g_i|,|g_{i+1}|)=1\};$$

for $S_i \subseteq G_i$, $i \in \{1, \dots, k\}$, we define

$$\varphi^*(S_1, \dots, S_k) = \varphi((S_1 \times \cdots \times S_k) \cap T).$$

The subgroups $G_1, \dots, G_k$ for which the definitions of φ and φ^* apply will be clear from the context.

For $i \in \{1, \dots, k\}$, let $X_i, Y_i \subseteq G_i$. Similar to [9], for $J \subseteq \{1, \dots, k\}$ we can define

$$\varphi_J(X_i; Y_i) = \varphi(Z_1, \dots, Z_k) \quad \text{with} \quad Z_i = \begin{cases} X_i & \text{if } i \in J, \\ Y_i & \text{if } i \notin J. \end{cases}$$

Notice that in order for φ_J to be well-defined, we just need the subsets X_i where $i \in J$ and the subsets Y_i where $i \notin J$. In a similar way, define

$$\varphi^*_J(X_i; Y_i) = \varphi((Z_1 \times \cdots \times Z_k) \cap T).$$

If $J = \{1, \dots, k\}$, by abuse of notation, we will just write $\varphi_J(X_i; Y_i) = \varphi(X_i)$ and $\varphi^*_J(X_i; Y_i) = \varphi^*(X_i)$.

Remark 3.5. Notice that whenever we have subgroups $G_1, \dots, G_\ell$ of a profinite group G with $G_j \leq N_G(G_i)$ for every $j \leq i$, and we take an open subgroup $U \trianglelefteq_o G_\ell$, there exists an open normal subgroup $V \trianglelefteq_o G$ such that $V \cap G_\ell \leq U$. This implies that $V \cap G_\ell \trianglelefteq G_1 \cdots G_\ell$ and $V \trianglelefteq_o G_\ell$.

The next two lemmas are useful applications of basic commutator calculus. The first one follows the ideas of Lemma 2.8 of [5] while Lemma 3.7 is an application of Lemma 3.6 to coprime commutators.

Lemma 3.6. *Let $G_1, \dots, G_k$ be subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for every $j \leq i$. For every $i \in \{1, \dots, k\}$ let $g_i \in G_i$, and for a fixed $\ell \in \{1, \dots, k\}$, let $g'_\ell \in G_\ell$. Then*

$$\varphi_{\{\ell\}}(g'_\ell g_\ell; g_i) = [g_1, \dots, g_{\ell-1}, g'_\ell, g_{\ell+1}^{h_\ell}, \dots, g_k^{h_{k-1}}]^{h_k} \varphi(g_i),$$

where $h_i \in G_\ell \cdots G_i$ for $i \in \{\ell, \dots, k\}$.

Proof. Assume first that $\ell \neq 1$, and proceed by induction on $k - \ell$. If $k - \ell = 0$, then

$$[g_1, \dots, g_{k-1}, g'_k g_k] = [g_1, \dots, g'_k]^{g_k [g_1, \dots, g_k]^{-1}} [g_1, \dots, g_k],$$

and the result follows. Assume $k - \ell > 0$, and we write, for the sake of brevity, $y = [g_1, \dots, g_{\ell-1}]$. By induction, we have

$$[y, g'_\ell g_\ell, g_{\ell+1}, \dots, g_k] = [[y, g'_\ell, g_{\ell+1}^{h_\ell}, \dots, g_{k-1}^{h_{k-2}}]^{h_{k-1}} [g_1, \dots, g_{k-1}], g_k]$$

with $h_i \in G_\ell \cdots G_i$ for $i \in \{\ell, \dots, k-1\}$. Now,

$$\begin{aligned} &[[y, g'_\ell, g_{\ell+1}^{h_\ell}, \dots, g_{k-1}^{h_{k-2}}]^{h_{k-1}} [g_1, \dots, g_{k-1}], g_k] \\ &\qquad = [[y, g'_\ell, g_{\ell+1}^{h_\ell}, \dots, g_{k-1}^{h_{k-2}}]^{h_{k-1}}, g_k]^{[g_1, \dots, g_{k-1}]} [g_1, \dots, g_k] \\ &\qquad = [y, g'_\ell, g_{\ell+1}^{h_\ell}, \dots, g_{k-1}^{h_{k-2}}, g_k^{(h_{k-1})^{-1}}]^{h_{k-1}[g_1, \dots, g_{k-1}]} [g_1, \dots, g_k], \end{aligned}$$

and the lemma follows. If $\ell = 1$, a similar argument applies. □

Lemma 3.7. *Let $G_1, \dots, G_k$ be subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for every $j \leq i$. Let $\ell \in \{1, \dots, k\}$ and $H_1, H_2 \subseteq G_\ell$ be such that $\pi(h_1), \pi(h_2) \subseteq \pi(h_1 h_2)$ for every $h_1 \in H_1$, $h_2 \in H_2$. Let $X_i \subseteq G_i$ for $i \in \{1, \dots, \ell - 1\}$, and for $i \in \{\ell + 1, \dots, k\}$ denote $X_i = G_i$. Then:*

(i) *If $\varphi^*_{\{\ell\}}(H_1; X_i) = \varphi^*_{\{\ell\}}(H_2; X_i) = 1$, then $\varphi^*_{\{\ell\}}(H_1 H_2; X_i) = 1$.*

(ii) *If* $\varphi^*_{\{\ell\}}(H_j;X_i)=\varnothing$ *for some* $j\in\{1,2\}$, *then* $\varphi^*_{\{\ell\}}(H_1H_2;X_i)=\varnothing$.

Proof. Since $\pi(h_1),\pi(h_2)\subseteq\pi(h_1h_2)$ for every $h_1\in H_1$, $h_2\in H_2$, the second statement is straightforward. Moreover, if $\varphi_{\{\ell\}}(h_1h_2;g_i)\in\varphi^*_{\{\ell\}}(H_1H_2;X_i)$, then for $j\in\{1,2\}$ we have $\varphi_{\{\ell\}}(h_j;g_i)\in\varphi^*_{\{\ell\}}(H_j;X_i)$. The result follows now directly from Lemma 3.6. □

In view of the preceding lemma, we now introduce a convenient way to choose coset representatives of normal subgroups. These will play an important role throughout the paper.

Definition 3.8. Let G be a profinite group and $U\trianglelefteq G$. An element $g\in G$ is a *good representative* of the coset gU if $\pi(g),\pi(u)\subseteq\pi(gu)$ for every $u\in U$.

Remark 3.9. In Definition 3.8, if G is pronilpotent, then any element $g\in G$ can be written in an unique way as $g=\prod_{p\in\pi(G)}g_p$ with g_p a p-element of G. Hence, g is a good representative of the coset gU if $g_p=1$ whenever $g_p\in U$ for $p\in\pi(G)$, or equivalently, if $\pi(g)$ is minimal among all representatives of the coset gU.

The following lemma is an application of Proposition 2.2 to a special type of coprime commutators.

Lemma 3.10. *Let* $G_1,\dots,G_k$ *be pronilpotent subgroups of a profinite group* G *such that* $G_j\le N_G(G_i)$ *for all* $j\le i$, G_k *abelian, and* $|G_{\delta^*_k}|<2^{\aleph_0}$. *For every* $i\in\{1,\dots,k\}$, *let* S_i *be a closed subset of* G_i. *If* $\varphi^*(S_i)\ne\varnothing$, *then, there exist elements* $x_i\in G_i$ *and open subgroups* $U_i\trianglelefteq_o G_i$ *such that* $|\varphi^*(x_iU_i\cap S_i)|=1$.

Proof. Let
$$T=\Big\{(x_1,\dots,x_k)\in S_1\times\cdots\times S_k\ \Big|\ (|x_i|,|x_{i+1}|)=1\text{ for all }i=1,\dots,k\Big\}.$$
If T is empty there is nothing to prove, so assume $T\ne\varnothing$. Note that T is closed in $G_1\times\cdots\times G_k$ by Lemma 2.7, and define the map
$$\begin{aligned}\varphi:T&\longrightarrow G_k\\ (x_1,\dots,x_k)&\longmapsto[x_1,\dots,x_k].\end{aligned}$$

We first prove, by induction on i, that $[x_1,\dots,x_i]\in G_{\delta^*_i}$ for every $i\in\{1,\dots,k\}$ and every $x_j\in G_j$ with $j\in\{1,\dots,i\}$. If $i=1$ the result is obvious, so assume $i>1$ and that $[x_1,\dots,x_{i-1}]$ is a δ^*_{i-1}-value. Then, by Lemmas 2.4 and 2.5 it follows that $[x_1,\dots,x_i]$ is a δ^*_i-value, as we wanted.

Hence, $|\operatorname{Im}(\varphi)|<2^{\aleph_0}$, and by Proposition 2.2, it follows that there exist elements $x_i\in G_i$ and open normal subgroups $U_i\trianglelefteq G_i$ such that
$$T\cap(x_1U_1\times\cdots\times x_kU_k)\ne\varnothing$$
and $|\varphi^*(x_iU_i\cap S_i)|=1$. □

Lemma 3.10 will often provide some cosets of open subgroups of G in which coprime commutators are trivial. Lemmas 3.11 and 3.13 below will allow us to relate coprime commutators of these cosets with coprime commutators of the open subgroups themselves.

Lemma 3.11. *Let* $G_1,\dots,G_k,K$ *be subgroups of a profinite group* G *such that* $G_j\le N_G(G_i)$ *for every* $j\le i$, *and for every* $i\in\{1,\dots,k\}$, *let* $x_i\in G_i$ *and* $U_i\trianglelefteq G_i$. *Assume also that* $G_j\le N_G(U_i)$ *for every* $j\le i$. *Fix* $j\in\{1,\dots,k\}$ *and write* $J=\{1,\dots,j-1\}$, *then:*

(i) *If* $\varphi(x_iU_i) = 1$ *then* $\varphi_J(x_iU_i; U_i) = 1$.
(ii) *If* $\varphi_J(x_iU_i; U_i) = 1$ *then*

$$\varphi_{J\cup\{j\}}(x_iU_i; U_i) = \varphi(x_1U_1, \ldots, x_{j-1}U_{j-1}, x_j, U_{j+1}, \ldots, U_k).$$

Proof. (i) We will proceed by reverse induction on $j \in \{1, \ldots, k+1\}$, where the base case $j = k+1$ translates to $\varphi(x_iU_i) = 1$, which is true by hypothesis. Let thus $j < k+1$ and assume that $\varphi_{J\cup\{j\}}(x_iU_i; U_i) = 1$.

Let $C_k = 1$ and for every $i \in \{j+1, \ldots, k-1\}$ define $C_i = C_{U_i}(U_{i+1}/C_{i+1})$. Note that C_i is well-defined, since using that for every ℓ the subgroup U_ℓ is normal in $G_1 \cdots G_\ell$, one can easily show by induction that $C_\ell \trianglelefteq G_1 \cdots G_\ell$.

If $j \geq 2$, let

$$Y = \{[x_1u_1, \ldots, x_{j-1}u_{j-1}] \mid u_i \in U_i,\ i = 1, \ldots, j-1\}.$$

Then, we can rewrite $\varphi_{J\cup\{j\}}(x_iU_i; U_i) = 1$ as

$$[Y, x_jU_j] \subseteq C_{G_j}(U_{j+1}/C_{j+1}).$$

For every $i \in \{1, \ldots, j\}$, fix $u_i \in U_i$ and shorten $y = [x_1u_1, \ldots, x_{j-1}u_{j-1}]$. Then we have $[y, x_ju_j] = [y, u_j][y, x_j]^{u_j}$, and since $C_{G_j}(U_{j+1}/C_{j+1})$ is a normal subgroup of G_j containing $[y, x_ju_j]$ and $[y, x_j]$, it follows that $[y, u_j] \in C_{G_j}(U_{j+1}/C_{j+1})$. This shows that $\varphi(x_1U_1, \ldots, x_{j-1}U_{j-1}, U_j, U_{j+1}, \ldots, U_k) = 1$, as we wanted.

For the case $j = 1$, note that both x_1 and x_1U_1 lay in $C_{G_1}(U_2/C_2)$, so that $U_1 \leq C_{G_1}(U_2/C_2)$.

(ii) For every $i \in \{j+1, \ldots, k\}$ we define C_i as in (i). For $i \in \{1, \ldots, k\}$, let $u_i \in U_i$ and shorten $y = [x_1u_1, \ldots, x_{j-1}u_{j-1}]$. Then,

$$[y, x_ju_j] = [y, u'x_j] = [y, x_j][y, u']^{x_j} = [y, u']^{x_i[x_j,y]}[y, x_j]$$

for some $u' \in U_j$, and note that $z := [y, u']^{x_j[x_j,y]} \in C_{G_j}(U_{j+1}/C_{j+1})$. Then,

$$[y, x_ju_j, u_{j+1}, \ldots, u_k] = [z[y, x_j], u_{j+1}, \ldots, u_k] = [y, x_j, u_{j+1}, \ldots, u_k],$$

where the last equality follows from Lemma 3.6. The lemma follows. □

Definition 3.12. Let $G_1, \ldots, G_k$ be pronilpotent subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for all $j \leq i$. Let σ be a finite set of primes. We define the normal subgroup

$$N_\sigma = \langle \varphi^*_{\{j\}}(H_i; G_i) \mid j \text{ is such that } |\pi(G_j)| = \infty \rangle^G,$$

where H_i is the Hall σ-subgroup of G_i for every i. If $|\pi(G_i)| < \infty$ for all i, then $N_\sigma = \langle \varnothing \rangle^G = 1$ for every σ.

The subgroups $G_1, \ldots, G_k$ of G for which the definition of N_σ applies will be clear from the context. Notice that for any finite sets of primes σ_1 and σ_2 such that $\sigma_1 \subseteq \sigma_2$ we have

$$N_{\sigma_1} \leq N_{\sigma_2}. \tag{3.1}$$

Lemma 3.13. *Let* $G_1, \ldots, G_k$ *be pronilpotent subgroups of a profinite group* G *such that* $G_j \leq N_G(G_i)$ *for all* $j \leq i$. *Fix* $\ell \in \{1, \ldots, k\}$. *For* $i \in \{1, \ldots, \ell-1\}$, *let* $X_i \subseteq G_i$, *and for* $i \in \{\ell, \ldots, k\}$ *let* $x_i \in G_i$ *and* $U_i \trianglelefteq_o G_j$ *be such that* $G_j \leq N_G(U_i)$ *for* $j \leq i$. *Suppose that* $(|x_\ell|, |x_{\ell-1}|) = 1$ *for every* $x_{\ell-1} \in X_{\ell-1}$. *Then:*

(i) *If* $(|x_\ell|, |U_{\ell+1}|) = 1$ *and* $\varphi^*(X_1, \ldots, X_{\ell-1}, x_\ell U_\ell, U_{\ell+1}, \ldots, U_k) = 1$, *then* $\varphi^*(X_1, \ldots, X_{\ell-1}, U_\ell, \ldots, U_k) = 1$.

(ii) *For $i \in \{\ell, \dots, k\}$, suppose that either $|\pi(G_i)| = \infty$, in which case we write $Y_i = G_i$, or $|\pi(G_i)| = 1$, in which case we write $Y_i = U_i$. If $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, \dots, x_k U_k) = 1$, then there exists a finite set of primes σ such that $\varphi^*(X_1, \dots, X_{\ell-1}, Y_\ell, \dots, Y_k) \subseteq N_\sigma$ (cf. Definition 3.12). Moreover, if the x_i are good representatives of the cosets x_iU_i (cf. Definition 3.8) such that $(|x_i|, |x_{i+1}|) = 1$ for all $i \in \{\ell, \dots, k-1\}$, then if $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, \dots, x_k U_k) = \varnothing$, also $\varphi^*(X_1, \dots, X_{\ell-1}, U_\ell, \dots, U_k) = \varnothing$.*

Proof. (i) Since $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, U_{\ell+1}, \dots, U_k) \neq \varnothing$, there are $x_1, \dots, x_{\ell-1}$ such that $x_i \in X_i$ and

$$(|x_j|, |x_{j+1}|) = 1 \tag{3.2}$$

for all $j \in \{1, \dots, \ell-2\}$. For $i \in \{\ell, \dots, k\}$, let H_i be a Hall π_i-subgroup of U_i, where we choose π_i such that

$$(|x_{\ell-1}|, |H_\ell|) = (|H_j|, |H_{j+1}|) = 1 \tag{3.3}$$

for all $j \in \{\ell, \dots, k\}$. Since G_ℓ is pronilpotent, we have $\pi(x_\ell h) \subseteq \pi(x_\ell) \cup \pi(h)$ for all $h \in H_\ell$, and hence, the set $\varphi(x_1, \dots, x_{\ell-1}, x_\ell H_\ell, H_{\ell+1}, \dots, H_k)$ is contained in $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, U_{\ell+1}, \dots, U_k)$, and so it is trivial. Lemma 3.11 now gives $\varphi(x_1, \dots, x_{\ell-1}, H_\ell, \dots, H_k) = 1$. Since this holds for all $x_1, \dots, x_{\ell-i}$ and for all Hall subgroups of $U_i, i \geq \ell$ satisfying (3.2) and (3.3) respectively, the claim follows.

(ii) Write $L = \{\ell, \dots, k\}$, and for $i \in L$, define

$$\sigma_i = \begin{cases} \pi(G_i/U_i) & \text{if } |\pi(G_i)| = \infty, \\ \pi(G_i) & \text{if } |\pi(G_i)| = 1. \end{cases}$$

Let $\sigma = \sigma_\ell \cup \dots \cup \sigma_k$, and assume that the x_i are all good representatives, and in particular that they are all σ-elements (cf. Remark 3.9). Furthermore, since $\varphi_L^*(x_iU_i; X_i) \neq \varnothing$, we may also assume that $(|x_i|, |x_{i+1}|) = 1$ for all $i \in \{\ell, \dots, k-1\}$. For $i \in L$ with $|\pi(G_i)| = \infty$, let V_i be the Hall σ'-subgroups of G_i, and for $i \in L$ with $|\pi(G)| = 1$, set $V_i = U_i$ (notice that $V_i \leq U_i$ if $|\pi(G_i)| = \infty$). Applying (i), we obtain $\varphi_L^*(V_i; X_i) = 1$, and from Lemma 3.7(i), we clearly obtain $\varphi_L^*(Y_i; X_i) \subseteq N_\sigma$, as desired.

Regarding the last claim, if $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, \dots, x_k U_k) = \varnothing$ then we also have $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell, \dots, x_k) = \varnothing$, so there exists an index $j \in \{1, \dots, l-2\}$ such that $(|x_j|, |x_{j+1}|) \neq 1$ for all $x_j \in X_j$, $x_{j+1} \in X_{j+1}$, and the lemma follows. ☐

The following lemma is the focal point of the proof of Proposition 3.16, as it will allow us to funnel some values of certain coprime commutators into an accurately chosen subgroup.

Lemma 3.14. *Let $G_1, \dots, G_k$ be pronilpotent subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for all $j \leq i$, G_k abelian, and $|G_{\delta_k^*}| < 2^{\aleph_0}$. Then, there exist a finite set $W \subseteq \varphi^*(G_i)$ and a finite set σ of primes such that $\varphi^*(G_i) \leq N_\sigma \langle W \rangle^G$.*

Proof. Let

$$\mathcal{I} = \{i \in \{1, \dots, k\} \mid |\pi(G_i)| = \infty\}.$$

Notice that it suffices to prove the lemma in the case when $|\pi(G_i)| = 1$ for all G_i with $i \notin \mathcal{I}$; the general case will follow by applying Lemma 3.7.

Also, for $i \notin \mathcal{I}$, let p_i be a prime such that $\pi(G_i) = \{p_i\}$. Then we have

$$\varphi^*(G_i) = \varphi^*(G_1, \dots, G_{i-2}, H_{i-1}, G_i, H_{i+1}, G_{i+2}, \dots, G_k),$$

where H_{i-1} and H_{i+1} are the Hall p_i'-subgroups of G_{i-1} and G_{i+1}, respectively. We can therefore assume, again by Lemma 3.7(i), that for all $i \notin \mathcal{I}$ we have

$$\pi(G_i) \not\subseteq \pi(G_{i-1}) \cup \pi(G_{i+1}). \tag{3.4}$$

We claim that that for every $J \subseteq \{1, \ldots, k\} \smallsetminus \mathcal{I}$ there exist a finite set $W_J \subseteq \varphi^*(G_i)$, a finite set of primes $\sigma(J)$ and subgroups $U_i \trianglelefteq_o G_i$ with $i \notin \mathcal{I} \cup J$ such that $\varphi^*_{\mathcal{I}\cup J}(G_i; U_i) \subseteq N_{\sigma(J)}\langle W_J\rangle^G$.

We proceed by induction on $|J|$. Assume first $J = \varnothing$. By Lemma 3.10, for every $i \in \{1, \ldots, k\}$ there exist elements $x_i \in G_i$ and subgroups $U_i \trianglelefteq_o G_i$ such that $\varphi^*(x_iU_i) = \{w_\varnothing\}$ for a suitable $w_\varnothing \in G$. Moreover, by Remark 3.5, we may assume that $G_j \leq N_G(U_i)$ for every $j \leq i$. Hence, Lemma 3.13(ii) produces a finite set $\sigma(\varnothing)$ of primes such that $\varphi^*_{\mathcal{I}}(G_i; U_i) \subseteq N_{\sigma(\varnothing)}\langle w_\varnothing\rangle^G$, so the claim follows for $|J| = 0$.

Assume now that $|J| \geq 1$ and that for every $J^- \subsetneq J$ there exist a finite set $W_{J^-} \subseteq \varphi^*(G_i)$, a finite set of primes $\rho(J^-)$ and subgroups $U_i^{J^-} \trianglelefteq_o G_i$, $i \notin \mathcal{I}$, such that $\varphi^*_{\mathcal{I}\cup J^-}(G_i; U_i^{J^-}) \subseteq N_{\rho(J^-)}\langle W_{J^-}\rangle^G$ (here we are taking $U_i^{J^-} = G_i$ if $i \in J^-$). Let $W_J = \bigcup_{J^-} W_{J^-}$, $\rho = \bigcup_{J^-} \rho(J^-)$ and $V_i = \bigcap_{J^-} U_i^{J^-}$, so that, by (3.1), we have $\varphi^*_{\mathcal{I}\cup J^-}(G_i; V_i) \subseteq N_\rho\langle W_J\rangle^G$ for every $J^- \subsetneq J$. Furthermore, by factoring out $N_\rho\langle W_J\rangle^G$, we may assume that

$$\varphi^*_{\mathcal{I}\cup J^-}(G_i; V_i) = 1 \tag{3.5}$$

for every $J^- \subsetneq J$. Moreover, taking into account Remark 3.5 we may further assume that V_i is invariant under the conjugacy action of G_j for every $j \leq i$.

Write $J = \{j_1, \ldots, j_n\}$ with $j_1 < \cdots < j_n$, and for every $i \in J$, fix a set S_i of coset representatives for V_i in G_i containing the identity. Write

$$S_{j_1} \times \cdots \times S_{j_n} = \{\mathbf{s}_1, \ldots, \mathbf{s}_m\}$$

with $\mathbf{s}_\ell = (s_{\ell,j_1}, \ldots, s_{\ell,j_n})$ for $\ell \in \{1, \ldots, m\}$. Denote $V_i = G_i$ for $i \in \mathcal{I}$. Since $1 \in S_i$ for every i, we have $\varphi^*_J(S_i, V_i) \neq \varnothing$, so applying Lemma 3.10 we obtain elements $x_i \in V_i$ and subgroups $U_i \trianglelefteq_o V_i$ such that $\varphi^*_J(S_i; x_iU_i)$ takes a single value. Actually, since $1 = \varphi^*_J(1, x_iU_i) \subseteq \varphi^*_J(S_i; x_iU_i)$, we have $\varphi^*_J(S_i; x_iU_i) = 1$. Also, we may assume the x_i to be good representatives of the U_i (cf. Definition 3.8) and therefore, if J does not contain neither i nor $i+1$, then $(|x_i|, |x_{i+1}|) = 1$. Thus, for every $\ell \in \{1, \ldots, m\}$, we either have

$$\varphi^*_J(s_{\ell,i}; x_iU_i) = \varnothing \quad \text{or} \quad \varphi^*_J(s_{\ell,i}; x_iU_i) = 1. \tag{3.6}$$

Again, by Remark 3.5 we may further assume that U_i is invariant under the conjugacy action of G_j for every $j \leq i$.

Let $J_0 = \varnothing$, and for $r \in \{1, \ldots, n\}$, let $J_r = \{j_1, \ldots, j_r\}$. We also write $j_0 = 0$ for convenience. We will show that for every $r \in \{0, \ldots, n\}$, there exists a finite set of primes $\sigma(r)$ such that $\varphi^*_{J_r}(s_{\ell,i}; Y_i^{(r)}) \subseteq N_{\sigma(r)}$ for every $\ell \in \{1, \ldots, m\}$, where

$$Y_i^{(r)} = \begin{cases} G_i & \text{if } i \geq j_r,\ i \in \mathcal{I} \cup J, \\ U_i & \text{if } i > j_r,\ i \notin \mathcal{I} \cup J, \\ x_iU_i & \text{if } i < j_r. \end{cases}$$

We argue by reverse induction on $r \in \{0, \ldots, n\}$; assume first $r = n$. Since $j_r \notin \mathcal{I}$, we deduce from (3.4) that $(|s_{\ell,j_r}|, |G_{j_r-1}|) = (|s_{\ell,j_r}|, |x_{j_r+1}|) = 1$. Thus, for all $\ell \in \{1, \ldots, m\}$, we obtain from (3.6) and Lemma 3.13(ii) a finite set of primes

$\sigma(r,\ell)$ such that $\varphi^*_{J_r}(s_{\ell,i};Y_i^{(r)}) \subseteq N_{\sigma(r,\ell)}$. Defining $\sigma(r) = \bigcup_{\ell=1}^m \sigma(r,\ell)$, we obtain $\varphi^*_{J_r}(s_{\ell,i};Y_i^{(r)}) \subseteq N_{\sigma(r)}$ for every $\ell \in \{1,\dots,m\}$.

Hence, we assume $r \leq n-1$. By induction, we know that there exists a finite set of primes $\sigma(r+1)$ such that

$$\varphi^*_{J_{r+1}}(s_{\ell,i};Y_i^{(r+1)}) \subseteq N_{\sigma(r+1)} \tag{3.7}$$

for every $\ell \in \{1,\dots,m\}$. We will first show that $\varphi^*_{J_r}(s_{\ell,i};Y_i^{(r+1)}) \subseteq N_{\sigma(r+1)}$. Note that $Y_i^{(r+1)} \leq V_i$ for every $i \notin \mathcal{I} \cup J$ and that $U_{j_{r+1}} \leq V_{j_{r+1}}$, so (3.5) yields

$$\varphi^*_{J_r}(s_{\ell,i};\widetilde{Y}_i) \subseteq N_{\sigma(r+1)}, \tag{3.8}$$

where $\widetilde{Y}_i = Y_i^{(r+1)}$ if $i \neq j_{r+1}$ and $\widetilde{Y}_{j_{r+1}} = U_{j_{r+1}}$. As we chose the sets of representatives S_j in such a way that the identity is contained in them, the element $s_{\ell,j_{r+1}}$ is a good representative for every $\ell \in \{1,\dots,m\}$. Thus, by (3.7) and (3.8), we deduce from Lemma 3.7 that $\varphi^*_{J_r}(s_{\ell,i};\overline{Y}_i) \subseteq N_{\sigma(r+1)}$, where $\overline{Y}_i = Y_i^{(r+1)}$ if $i \neq j_{r+1}$ and $\overline{Y}_{j_{r+1}} = s_{\ell,j_{r+1}}U_{j_{r+1}}$. Since this holds for every $\ell \in \{1,\dots,m\}$, and since $G_{j_{r+1}} = \bigcup_{s\in S_{j_{r+1}}} sU_{j_{r+1}}$, we obtain $\varphi^*_{J_r}(s_{\ell,i};Y_i^{(r+1)}) \subseteq N_{\sigma(r+1)}$, as we wanted.

Now using (3.4) and Lemma 3.13, we conclude exactly as in the case $r = n$ that there exists a finite set $\sigma(r)$ of primes such that $\varphi^*_{J_r}(s_{\ell,i};Y_i^{(r)}) \subseteq N_{\sigma(r)}$ for every $\ell \in \{1,\dots,m\}$.

This completes the reverse induction on r. In particular, for $r = 0$, it follows that $\varphi^*_J(G_i;U_i) \subseteq N_{\sigma(0)}$, so this, in turn, concludes the inductive step on $|J|$, and the claim is proved.

Finally, taking J in such a way that $\mathcal{I} \cup J = \{1,\dots,k\}$, we obtain a finite set of primes $\sigma(J)$ and a finite set $W \subseteq \varphi^*(G_i)$ such that $\varphi^*(G_1,\dots,G_k) \subseteq N_{\sigma(J)}\langle W\rangle^G$, as desired. □

Recall that if G is a profinite group of Fitting height $k+1$, then there exist pronilpotent subgroups $T_1,\dots,T_k$ of G such that $T_j \leq N_G(T_i)$ for every $j \leq i$. Moreover, $G = T_1 \cdots T_k$, $T_k = \delta_k^*(G)$ and $T_i = [T_1,\dots,T_i]$ for every $i \in \{1,\dots,k\}$ (see the beginning of Section 3.2).

Lemma 3.15. *Let $G = T_1 \cdots T_k$ be as above with $T_k = \delta_k^*(G)$ abelian, and assume $|G_{\delta_k^*}| < 2^{\aleph_0}$. Let $g \in \varphi^*(T_i)$. Then, there exists a finite normal subgroup $N \trianglelefteq G$ such that $g \in N$.*

Proof. Write $g = [x_1,\dots,x_k]$, where $x_j \in T_j$ for all j and $(|x_\ell|,|x_{\ell+1}|) = 1$ for all $\ell \in \{1,\dots,k-1\}$. In the same way as in the proof of Lemma 3.10, we see that $[x_1,\dots,x_i]$ is a δ_i^*-value for every $i \in \{1,\dots,k\}$.

In particular $x := [x_1,\dots,x_{k-1}]$ is a δ_{k-1}^*-value, and let H be the minimal Hall subgroup of $\delta_k^*(G)$ containing x_k, so that $(|x|,|H|) = 1$. Since, again, $[x,h]$ is a δ_k^*-value for every $h \in H$, the set $K := \{[x,h] \mid h \in H\}$ has less than $2^{\aleph_0}$ values, and, since H is abelian and normal in G, it follows that K is actually a subgroup of G. In particular, K is finite, so every element of K has finite order. Thus, we deduce from Lemma 2.3 that the set $T := \bigcup\{k^G \mid k \in K\}$ is finite, and therefore $N = \langle T\rangle$ is finite. □

We are now ready to prove the strong conciseness of δ_k^* in prosoluble groups of Fitting height $k+1$.

Proposition 3.16. *Let G be a prosoluble group of Fitting height $k+1$. Assume that $|G_{\delta_k^*}| < 2^{\aleph_0}$. Then $\delta_k^*(G)$ is finite.*

Proof. In view of Lemma 3.1, we may assume that $\delta_k^*(G)$ is abelian. Thus, we can take $T_1, \dots, T_k \leq G$ as in Lemma 3.15, so that $G = T_1 \cdots T_k$ with $T_k = \delta_k^*(G)$ abelian.

We claim that for every family of subgroups $G_i \leq T_i$ with $i \in \{1, \dots, k\}$ such that $G_j \leq N_G(G_i)$ for $j \leq i$, we have $|\varphi^*(G_i)| < \infty$. We argue by induction on $|\mathcal{I}|$, where

$$\mathcal{I} = \{i \in \{1, \dots, k\} \mid |\pi(G_i)| = \infty\}.$$

If $|\mathcal{I}| = 0$, then Lemma 3.14 gives the result since for every finite set $W \subseteq \varphi^*(G_i)$, the normal subgroup $\langle W \rangle^G$ is finite by Lemma 3.15, and since, by definition, $N_\sigma = 1$ for every finite set of primes σ. Suppose thus $|\mathcal{I}| \geq 1$. Then, Lemma 3.14 produces a finite set of primes σ and a finite set $W \subseteq \varphi^*(G_i)$ such that $|\varphi^*(G_i)| < N_\sigma \langle W \rangle^G$. Observe that by induction, for every $j \in \mathcal{I}$, we have $|\varphi^*_{\{j\}}(H_i; G_i)| < \infty$, where H_i is the Hall σ-subgroup of G_i, and therefore N_σ is finite by Lemma 3.15. Again by Lemma 3.15, $\langle W \rangle^G$ is also finite, and the claim follows.

In particular, we have shown that $|\varphi^*(T_i)| < \infty$. Now, for every $i \in \{1, \dots, k\}$ and every $p \in \pi(G)$, let $P_i(p)$ be the Sylow p-subgroup of T_i. Then, for every $i \in \{2, \dots, k\}$ and every $p \in \pi(G)$, Lemma 2.6 gives

$$P_i(p) = \prod_{\substack{q \in \pi(G) \\ q \neq p}} [P_{i-1}(q), P_i(p)],$$

and therefore, for every $q_k \in \pi(G)$,

$$P_k(q_k) = \prod_{(q_1, \dots, q_{k-1}) \in S_{q_k}} [P_1(q_1), \dots, P_k(q_k)],$$

where

$$S_{q_k} = \{(q_1, \dots, q_{k-1}) \in \pi(G)^{k-1} \mid q_i \neq q_{i+1} \text{ for every } i = 1, \dots, k-1\}.$$

By Lemma 3.6, this implies that $P_k(q_k) \leq \langle \varphi^*(T_i) \rangle$ for every $q_k \in \pi(G)$, and so

$$\delta_k(G) = T_k = \prod_{p \in \pi(G)} P_k(p) = \langle \varphi^*(T_i) \rangle.$$

The proposition follows from Lemma 3.15. □

4. The main theorems

We recall that a *minimal simple group* is a finite non-abelian simple group all of whose proper subgroups are soluble. These groups have been classified by Thompson in [26].

Lemma 4.1. *In every minimal simple group there exist an involution e and an element h of odd order such that $h^e = h^{-1}$. Moreover, for every positive integer k, the element*

$$g_k = [h, e, \overset{k-1}{\ldots}, e]$$

is both a non-trivial γ_k^*-value and a non-trivial δ_{k-1}^*-value.

Proof. The first claim follows from Theorem 2.13 of [16] and the fact that non-abelian simple groups are of even order. Notice that $g_k = h^{(-2)^{k-1}}$, so $g_k \neq 1$ for every positive integer k. Clearly g_i and e are coprime for every $i \in \{1, \dots, k-1\}$, so g_k is a γ_k^*-value. Thus, it suffices to prove that the same is true for δ_{k-1}^*. By Proposition 25 of [4], every involution of a minimal simple group is a δ_ℓ^*-value for every $\ell \in \mathbb{N}$. Hence, we can use Lemma 2.5 with $y_1 = \cdots = y_{k-1} = e$ and $H = \langle h \rangle$ and conclude the proof. □

We are now ready to prove our main results. As in [10], we start showing that the Fitting subgroup of any profinite group G with $|G_{\gamma_k^*}| < 2^{\aleph_0}$ or $|G_{\delta_k^*}| < 2^{\aleph_0}$ is infinite.

Proposition 4.2. *Let G be an infinite profinite group and let $w^* = \delta_k^*$ or $w^* = \gamma_k^*$. Suppose that $|G_{w^*}| < 2^{\aleph_0}$. Then the Fitting subgroup F of G is infinite.*

Proof. We first show that F is non-trivial. Assume by contradiction that $F = 1$. For every w^*-value x of G, the normal closure $\langle x^G \rangle$ is finite. Indeed, by Lemma 2.3, x^G is finite, so in particular $|G : C_G(\langle x^G \rangle)| < \infty$. As a consequence, the index $|\langle x^G \rangle : Z(\langle x^G \rangle)|$ is also finite, but $Z(\langle x^G \rangle)$ is contained in $F = 1$. This implies that G possesses finite minimal normal subgroups, so let N be the product of all of them.

If N is finite, then there exists a normal open subgroup $K \trianglelefteq_o G$ such that $K \cap N = 1$. Such a subgroup cannot contain any w^*-value since otherwise, repeating the same argument as before, we would obtain a minimal normal subgroup of G contained in K, contradicting that $K \cap N = 1$. If $w^*(K) = 1$, then by Theorem 2.1 K is either pronilpotent (if $w^* = \gamma_k^*$) or prosoluble of Fitting height k (if $w^* = \delta_k^*$), and this contradicts the fact that $F \cap K = 1$. This proves that N is an infinite subgroup of G.

None of the infinitely many minimal normal subgroups of G is abelian because $F = 1$, so each of these minimal normal subgroup contains a section isomorphic to a minimal simple group. For each minimal normal subgroup N_i, with $i \in I$, choose a section isomorphic to a minimal simple group S_i. We remark that by the previous discussion I is an infinite set and so the Cartesian product of S_i is a section of G. By Lemma 4.1 in each of these groups S_i there exist an involution $e_{S_i} \in S_i$ and an element $h_{S_i} \in S_i$ of odd order with $h_{S_i}^{e_{S_i}} = h_{S_i}^{-1}$ such that $g_{S_i} := [h_{S_i}, e_{S_i}, \overset{k-1}{\dots}, e_{S_i}]$ is a non-trivial w^*-value. Now, for each subset $J \subseteq I$, the element $c_J = \prod_{j \in J} g_{S_j}$ can be written as

$$c_J = \Big[\prod_{j \in J} h_{S_j}, \prod_{j \in J} e_{S_j}, \dots, \prod_{j \in J} e_{S_j}\Big].$$

Clearly $\prod_{j \in J} e_{S_j}$ is an involution normalizing the cyclic subgroup generated by the element of odd order $\prod_{j \in J} h_{S_j}$, and hence it is a w^*-value (if $w^* = \delta_k^*$ we also need to use Lemma 2.5). However, there exist at least $2^{\aleph_0}$ distinct subsets $J \subseteq I$ that give rise to different c_J, against the assumption that $|G_{w^*}| < 2^{\aleph_0}$, so $F \neq 1$.

If we assume by contradiction that the Fitting subgroup F is finite, then there would be a subgroup $K \trianglelefteq_o G$ with $K \cap F = 1$, so that K has trivial Fitting subgroup. By the previous argument, this can happen only if $w^*(K) = 1$, so K is either pronilpotent or prosoluble of Fitting height k, contradicting that $K \cap F = 1$ and proving the proposition. □

Proofs of Theorems 1.1 and 1.2. In view of Theorem 2.1 it is sufficient to show that G is finite-by-pronilpotent in the case $w^* = \gamma_k^*$ or finite-by-(prosoluble of

Fitting height k) if $w^* = \delta_k^*$. We can assume G to be infinite, otherwise the theorem is trivially true.

We will denote the Fitting subgroup of G by F, and for $k \geq 2$, let F_k be the k-th Fitting subgroup of G. By Proposition 4.2, F is infinite (and hence the same is true for all F_k). Let $t = 2$ if $w^* = \gamma_k^*$ and $t = k+1$ if $w^* = \delta_k^*$. By Propositions 3.4 and 3.16, $w^*(F_t)$ is finite. Therefore there exits an open normal subgroup $R \trianglelefteq_o F_t$ with $R \cap w^*(F_t) = 1$. Theorem 2.1 implies that the Fitting height of R is at most $t-1$, and hence R is contained in F_{t-1}. However, since F_t/F_{t-1} is the Fitting subgroup of G/F_{t-1}, it follows that G/F_{t-1} has finite Fitting subgroup, and by Proposition 4.2 this can only happen if G/F_{t-1} is finite.

Thus, we will prove the result by induction on $|G : F_{t-1}(G)|$, with the base case $G = F_{t-1}(G)$ being trivial. Assume then that $|G : F_{t-1}(G)| > 1$ and suppose first that G/F_{t-1} has a nontrivial proper normal subgroup N. The inductive hypothesis yields $|w^*(N)| < \infty$, and working in $G/w^*(N)$, we obtain by Theorem 2.1 that $N/w^*(N)$ is prosoluble of Fitting height at most $t-1$. This implies that $N/w^*(N)$ is contained in the $(t-1)$-th Fitting subgroup of $G/w^*(N)$ and by inductive hypothesis $w^*(G/w^*(N))$ must be finite, so $w^*(G)$ is finite too.

We can hence assume that G/F_{t-1} is a simple group. Notice that if G/F_{t-1} is abelian, then we can conclude simply by applying Proposition 3.4 or Proposition 3.16. Thus, the only case left is when G/F_{t-1} is a finite non-abelian simple group. By Theorem 2.1 we have $w^*(G/F_{t-1}) = G/F_{t-1}$, so there is a finite set S consisting of w^*-values such that $G = \langle S \rangle F_{t-1}$. By Lemma 2.3 the set $T := \bigcup\{s^G \mid s \in S\}$ is finite, so the index $|G : C_G(T)|$ is also finite. This implies that the center of $\langle T \rangle$ has finite index in $\langle T \rangle$, so by Schur's theorem $\langle T \rangle'$ is finite too. Note that $\langle T \rangle'$ is normal in G. Factoring out $\langle T \rangle'$, we can assume G/F_{t-1} to be abelian, and we conclude the proof as before. $\square$

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